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Complete Independence of Clones in the Ranked Pairs Rule

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Abstract. The "ranked pairs" voting rule introduced in Tideman [2] is independent of clones (not materially affected by the replication of a candidate) in all but a small domain of cases. Appending a particular tie-breaking rule to the ranked pairs rule generates a rule that is completely independent of clones.

I. Introduction

Tideman [2] introduced the concept of "independence of clones" as a criterion for voting rules. He offered a voting rule, the "ranked pairs rule," that was shown to be independent of clones in at least all but a small domain of cases, and to possess the properties of Condorcet consistency, non-negative responsiveness and resolvability. Whether the ranked pairs rule was independent of clones in all cases remained an open question. The purposes of this paper are to give the concept of independence of clones a more formally rigorous definition, to explain in greater detail why independence of clones is an attractive property, to show that the ranked pairs rule as originally defined is not independent of clones in all cases, and to show that a slight modification of the way that ties are handled in the ranked pairs rule yields a rule that is completely independent of clones.

II. Definitions

Let D denote a set of possible candidates. Let A (an agenda) denote a subset of those candidates that voters have "ranked."

A "ranking" is a transitive, asymmetric binary relation on an agenda, where transitivity and asymmetry are defined as follows:

Transitivity: $\forall x, y, z \in A: (xRy \,\&\, yRz) \Rightarrow (xRz)$.

Asymmetry: $\forall x, y \in A: (xRy) \Rightarrow \sim(yRx)$.

A complete ranking is a ranking that also satisfies the following condition:

Completeness: $\forall x, y \in A$: $(x \neq y) \Rightarrow (xRy \vee yRx)$.

If R is a ranking and neither xRy nor yRx, then x and y will be said to be tied according to R. A "profile" is a sequence of rankings. A "voting rule" is a function, V, that, for any A with $|A| \geqslant 2$ and any profile s of rankings of A, assigns a non-empty subset of A, $V(A, s)$. Let I denote a set of voters who have ranked the candidates in A.

Definiton 1. A subset of A, $C(s)$, is a *set of clones for the profile s* if and only if:

$2 \leqslant |C(s)| < |A|$, and

$\forall y \in C(s)$, $\forall z \in A \backslash C(s)$, $\forall i \in I$: $yR_i z \vee zR_i y$, and

$\forall x, y \in C(s)$, $\forall z \in A \backslash C(s)$, $\forall i \in I$:

$xR_i z \Rightarrow yR_i z$ and $zR_i x \Rightarrow zR_i y$.

There is a connection between the concept of clones and the concept of "a segment of a ranking" defined by Kemeny and Snell [1, p 10]. They define a set of objects, S, to form a segment of a ranking if the complement of S is not empty and if every element in the complement is either ahead of every element of S or behind every element of S. In terms of segments, $C(s)$ is a set of clones for the profile s if and only if $2 \leqslant |C(s)|$ and $C(s)$ is a segment of every ranking in s.

Let $s \backslash x$ denote the profile that is obtained when candidate x is removed from every ranking in profile s.

Definition 2. For any voting rule, V, *clone choice is independent of clone addition and deletion* if and only if $\forall$ agendas $A \subset D$, $\forall$ profiles s, $\forall$ sets of clones $C(s)$ and $\forall x \in C(s)$:

$$C(s) \cap V(A, s) \neq \emptyset \Leftrightarrow C(s) \backslash x \cap V(A \backslash x, s \backslash x) \neq \emptyset \ .$$

Definition 3. For any voting rule, V, *non-clone choice is independent of clone addition and deletion* if and only if $\forall$ agendas $A \subset D$, $\forall$ profiles s, $\forall$ candidates x, y such that x is a clone but x is not a clone of y:

$$y \in V(A, s) \Leftrightarrow y \in V(A \backslash x, s \backslash x)$$

Definition 4. A voting rule is *independent of clones* if and only if clone choice is independent of clone addition and deletion and non-clone choice is independent of clone addition and deletion.

III. Rationales

The rationale for calling $C(s)$ clones when the definition of clones is satisfied is connected to a spatial model of elections. If voters have preferences over some "attri bute space" in which candidates have locations, and all voters perceive two or more candidates to occupy the same location in attribute space, then there will be no voter who ranks any candidate between candidates who occupy the same location. Conversely, if three candidates are perceived to occupy different locations in attribute space, there will be some distance-related preferences for which each candidate is ranked between the other two. Any particular case of no candidate ever separating a set of other candidates could arise simply because no voter happened to have the preferences that would yield such a ranking, with the set of candidates that was never separated occupying different locations in attribute space. Nevertheless, the fact that a set of candidates is never separated by other candidates is consistent with the hypothesis that the candidates who are never separated are arbitrarily close to one another in attribute space. Thus it is reasonable to call a set of candidates satisfying the definition "clones".

The rationale for seeking independence of clones in voting rules is related to the spatial interpretation of clones. Given a set of locations of available candidates, one would expect an attractive voting rule not to let the location of the selected candidate be affected by the number of candidates from each location that were on the agenda. Even if all voters rank x ahead of y, if candidates x and y are clones and if x is the candidate that deserves to be selected from an agenda that includes x and y, then, if x is not available, y should be selected.

Independence of clones may be viewed as a weakening of choice consistency across agendas. Choice consistency across agendas would be attractive, but the possibility of cycles guarantees that no practical voting rule can achieve it. Therefore its weakening to independence of clones, which some practical voting rules satisfy and others do not, is an attractive criterion by which to evaluate voting rules.

IV. The Ranked Pairs Rule

It was shown in Tideman [2] that among a wide variety of previously proposed voting rules, there was none that was independent of clones and also possessed the properties of Condorcet consistency, non-negative responsiveness and resolvability. However, a new voting rule, the "ranked pairs" rule, was shown to possess these properties and, in almost all circumstances, to be independent of clones.

The ranked pairs rule can be defined algorithmically as follows: Define $M(x, y)$ as the difference between the number of voters who rank x ahead of y and the number who rank y ahead of x. Given voters' rankings of an agenda, A, consider the set of *unordered* pairs of distinct candidates in A. Define a ranking, T, of that set of pairs, by the rule that $\{x, y\}\, T \{u, v\}$ if and only if $|M(x, y)| > |M(u, v)|$. If $|M(x, y)| = |M(u, v)|$ then $\{x, y\}$ is tied with $\{u, v\}$ according to T. For $x, y \in A$ with $M(x, y) > 0$, and P a complete ranking of the *candidates* in A, define P to "describe" the pair $\{x, y\}$ if and only if xPy.

To implement the algorithm for the ranked pairs rule, consider all possible complete rankings of A. Eliminate the complete rankings that do not describe the first and second pairs in T. When one reaches the third and subsequent pairs, it is possible that none of the remaining complete rankings describe that pair. In that case, ignore that pair and proceed to the next. Continue until just one complete ranking of the candidates remains. The winner under the ranked pairs rule is the candidate at the top of that ranking.

If there are one or more ties in T then, whenever q of the pairs are tied with the same majority and none of the remaining complete rankings describe all of the tied pairs, consider each of the $q!$ ways of breaking the tie. For each way of breaking that and any subsequent ties, there will be a final complete ranking of the candidates. The election is a tie among all candidates that are at the top of a complete ranking that the algorithm generates for some way of breaking the ties among pairs. If the majority for any pair is 0 and the algorithm proceeds to the point where majorities of 0 would be considered, then all remaining complete rankings are winning complete rankings, and all candidates that head them are winning candidates.

In this paper we introduce a way of breaking ties under the ranked pairs rule, using the ranking of one voter (the tie-breaker). If the tie-breaker submits a ranking of the candidates that includes some ties, then a random process is used to resolve these, so that a complete ranking is produced. The resulting complete ranking of the candidates will be called a TBRC, or tie-breaking ranking of candidates. The TBRC serves to specify, for all pairs with $M(x,y)=0$, which of the two candidates in the pair is to be treated as if it had a positive majority over the other. From the TBRC one can derive, in a way to be specified, a TBRP, or tie-breaking ranking of the unordered pairs of candidates.

A TBRP is said to be "impartial" if and only if, for any $w, x, y, z \in A$, it ranks $\{w,y\}$ ahead of $\{x,y\}$ whenever it ranks $\{w,z\}$ ahead of $\{x,z\}$. An impartial TBRP can be constructed from a TBRC by ranking the elements in each pair according to which is ahead of the other in the TBRC, then ranking the pairs according to the TBRC's ranking of their first elements, and finally ranking pairs with the same first element according to the TBRC's ranking of their second elements. A TBRP so constructed is impartial because it ranks $\{w,y\}$ ahead of $\{x,y\}$ if and only if the TBRC ranks w ahead of x.

The circumstances in which it was not shown in Tideman [2] whether the ranked pairs rule is independent of clones were when there was some pair of candidates for which $M(x,y)=0$ or when the ranking of pairs contained ties involving candidates that were not clones. One might think that it would be possible to modify Tideman's proof and show that the ranked pairs rule is independent of clones in all cases. However, by providing a counterexample we show that this is not so. We then show that if ties are resolved by an impartial TBRP, then the ranked pairs rule is completely independent of clones.

V. A Counter-Example to the Independence of Clones of the Ranked Pairs Rule

Consider an election with seven candidates, a, b, b', c, d, e, and f (b and b' being clones), and nine voters, with rankings as shown by the columns in Example 1.

Example 1

d	e	b	c	d	a	a	f	f
e	b	b'	f	c	b'	c	e	e
b	b'	f	a	a	b	b'	a	b'
b'	f	c	d	e	c	b	c	b
f	c	a	e	b	d	d	b'	d
a	a	d	b	b'	e	e	b	c
c	d	e	b'	f	f	f	d	a

For Example 1, $M(x,y)$ is given by the following table:

$y=$	a	b	b'	c	d	e	f
$x=a$	...	1	1	−1	3	1	−3
b	−1	...	1	1	3	−3	3
b'	−1	−1	...	1	3	−3	3
c	1	−1	−1	...	3	1	−1
d	−3	−3	−3	−3	...	3	−1
e	−1	3	3	−1	−3	...	1
f	3	−3	−3	1	1	−1	...

To apply the ranked pairs rule, one first examines the pairs with a majority of 3. There are ten of these: $\{a,d\}$, $\{a,f\}$, $\{b,d\}$, $\{b,e\}$, $\{b,f\}$, $\{b',d\}$, $\{b',e\}$, $\{b',f\}$, $\{c,d\}$, and $\{d,e\}$. The relations specified by the majorities in these pairs are shown in Fig. 1, where $x \to y$ means that $M(x,y)=3$.

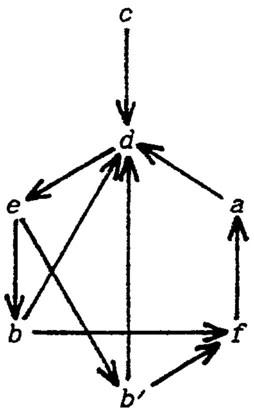

Fig. 1. Majorities of 3 in Example 1

It is possible for a to win in Example 1, by using a TBRP that ranks pairs $\{a,d\}$, $\{b,d\}$, $\{c,d\}$, $\{d,e\}$, $\{b',e\}$, and $\{b',f\}$ ahead of the other four pairs with $M(x,y)=3$. This places candidates a, b, and c ahead of all other candidates. If the TBRP also ranks pairs $\{a,b\}$ and $\{b,c\}$ ahead of $\{a,c\}$, then a is the final winner. However, if the clone b' is eliminated, then it is no longer possible for a to win. For a to win with b' absent, it would be necessary for pair $\{a,f\}$ to be ranked by the TBRP below the other four pairs in the cycle among d, e, b, f, and a. In addition, $\{b,d\}$ would have to be ranked below $\{b,e\}$ and $\{d,e\}$, to keep f from being ranked ahead of a. These conditions leave only two complete rankings: (a,c,d,e,b,f) and (c,a,d,e,b,f). Since $M(c,a)=1$, ranking (c,a,d,e,b,f) is the one that will be selected by the ranked pairs rule. Since a can win in Example 1 when b' is present but not when b' is absent, the ranked pairs rule is not independent of clones.

VI. A New Definition of the Ranked Pairs Rule

It was shown in Tideman [2] that the ranked pairs rule can be defined not only by the algorithm discussed above, but also as a function. For our purposes, it is useful to employ a third definition of the ranked pairs rule. Given a complete ranking of the candidates, P, let candidate x "attain" candidate y through P if and only if there is a sequence of distinct candidates, $a_1, a_2, \ldots, a_j$, with $a_1 = x$ and $a_j = y$, such that $a_i P a_{i+1}$ and $M(a_i, a_{i+1}) \geqslant M(a_j, a_1)$ for $i = 1, \ldots, j-1$. Let a complete ranking of the candidates, P, be a "stack" if and only if xPy implies x attains y through P. A candidate wins according to the third definition of the ranked pairs rule if and only if it is ranked first in a stack.

If P is a complete ranking the algorithm generates, then P is a stack and xPy implies that x attains y through P. The reason for this is that at the first point in the algorithm at which all complete rankings that rank y ahead of x have been eliminated, there must be a sequence of distinct candidates, $a_1, a_2, \ldots, a_j$, with $a_1 = x$ and $a_j = y$, such that $a_i P a_{i+1}$ for $i = 1, \ldots, j-1$. If $M(a_j, a_1)$ were greater than some $M(a_i, a_{i+1})$, then one or more rankings that ranked y ahead of x would have been retained. Therefore $M(a_i, a_{i+1}) \geqslant M(a_j, a_1)$ for $i = 1, \ldots, j-1$. In other words, a complete ranking is selected by the algorithmic definition of the ranked pairs rule only if it is a stack, and a candidate wins under the algorithmic definition of the ranked pairs rule only if it wins under the definition in terms of stacks.

If a candidate wins according to the definition of the ranked pairs rule in terms of stacks, then select as a TBRC a stack, P, that ranks that candidate first. Choose a TBRP that ranks every pair such that P describes that pair ahead of every pair such that P does not describe the pair. Such a TBRP produces P. Hence, if a candidate wins according to the definition of the ranked pairs rule in terms of stacks, then it also wins under the algorithmic definition. Thus the two definitions are equivalent.

VII. The Complete Independence of Clones of the Ranked Pairs Rule with an Impartial Tie-Breaker

Let A be an agenda. Let $C \subset A$ be a set of clones. Let c be an element of C that wins when the ranked pairs rule is applied just to C, and let P' be a stack for C that ranks c first. Let $B = (A \backslash C) \cup \{c\}$. Let z be a candidate that wins when the ranked pairs rule is applied just to B, and let P'' be a stack for B that ranks z first. Define P, a complete ranking of A, such that for every u and v that are distinct elements of C, and every x and y that are distinct elements of $A \backslash C$, the following four conditions hold:

(a) uPv if and only if $uP'v$,
(b) uPy if and only if $cP''y$,
(c) xPv if and only if $xP''c$,
(d) xPy if and only if $xP''y$.

Checking the four cases separately, one observes that P is a stack. Hence, z is a winner when the ranked pairs rule is applied to A. In other words, if z wins among B, then it also wins among A.

On the other hand, if an impartial TBRP is used under the ranked pairs rule, then in no case will an element of $A \backslash C$ be ranked between two elements of C in a winning complete ranking. Supposing to the contrary, that the ranked pairs rule selects a complete ranking $(\ldots, c_1, b_1, b_2, \ldots, b_k, c_2, \ldots)$, where c_1 and c_2 are elements of C, but $b_1, b_2, \ldots, b_k$ are not, let $\{b_i, c_2\}$ be the first pair of the form $\{b_j, c_2\}$ to be considered by the algorithm and selected by the TBRP. Since the TBRP is impartial, $\{c_1, b_i\}$ is the first pair of the form $\{c_1, b_j\}$ to be considered by the algorithm and selected by the TBRP. Because $\{b_i, c_2\}$ and $\{c_1, b_i\}$ are ranked ahead of other pairs of the forms $\{b_j, c_2\}$ and $\{c_1, b_j\}$ respectively, for any b_j in $A \backslash C$, $M(b_i, c_2) \geqslant M(b_j, c_2)$ and $M(c_1, b_i) \geqslant M(c_1, b_j)$. Since c_1 and c_2 are clones, $M(b_i, c_2) = -M(c_1, b_i)$. In order for b_i to be ranked ahead of c_2 and for c_1 to be ranked ahead of b_i, $(b_i, c_2) = 0 = M(c_1, b_i)$. Hence, the TBRC ranks b_i ahead of c_2 and c_1 ahead of b_i, which is a contradiction, since c_1 and c_2 are clones. Therefore, in no case will an element of $A \backslash C$ be ranked between two elements of C in P. All the elements of C will be together in P.

Consider a stack, P, in which all the elements of C are together. Define P', a complete ranking of C, and P'', a complete ranking of B, such that for every u and v that are distinct elements of C, and every x and y that are distinct elements of $A \backslash C$, the following four conditions hold:

(a) $uP'v$ if and only if uPv,
(b) $cP''y$ if and only if uPy,
(c) $xP''c$ if and only if xPv,
(d) $xP''y$ if and only if xPy.

Consider two elements of $A \backslash C$, x and y, such that x attains y through P. By definition, there is a sequence of distinct candidates, $a_1, \ldots, a_j$, with $a_1 = x$ and $a_j = y$ such that $a_i P a_{i+1}$ and $M(a_i, a_{i+1}) \geqslant M(a_j, a_1)$ for $i = 1, \ldots, j-1$. Because the elements of C in any subsequence of P are together, if the elements of C in the sequence $a_1, \ldots, a_j$ are collectively replaced by c, to yield a new sequence, $b_1, \ldots, b_k$, with $b_1 = x$ and $b_k = y$, then $b_i P b_{i+1}$ and $M(b_i, b_{i+1}) \geqslant M(b_k, b_1)$ for $i = 1, \ldots, k-1$. Therefore, if x, an element of $A \backslash C$, attains y, an element of $A \backslash C$, through P, then x attains y in B through P''. Effectively the same argument holds for elements of C, so that P' is a stack. Moreover, P' is a stack that ranks c first, and if z is ranked first by P, then z is also ranked first by P''.

To summarize, z is ranked first by a stack, P, which groups all the elements of C together, if and only if z is ranked first by stack P'. Similarly, if c is *any* clone, z is ranked first by P' if and only if z is ranked first among $A \backslash c$ by a stack, Q, which groups all the elements of $C \backslash c$ together. Thus, since the ranked pairs rule with an impartial TBRP groups the elements of any set of clones together, it is independent of clones.

References

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